

MODULAR REPRESENTATIONS AND BRANCHING RULES FOR AFFINE AND CYCLOTOMIC YOKONUMA-HECKE ALGEBRAS

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ABSTRACT. We give an equivalence between a module category of the affine Yokonuma-Hecke algebra (associated with the group $\mathbb{Z}/r\mathbb{Z}$) and its suitable counterpart for a direct sum of tensor products of affine Hecke algebras of type A . We then develop several applications of this result. In particular, the simple modules of the affine Yokonuma-Hecke algebra and of its associated cyclotomic algebra are classified over an algebraically closed field of characteristic p when p does not divide r . The modular branching rules for these algebras are obtained, and they are further identified with crystal graphs of integrable modules for quantum affine algebras.

1. INTRODUCTION

1.1. The modular branching rules for the symmetric groups \mathfrak{S}_n over an algebraically closed field \mathbb{K} of characteristic p were discovered by Kleshchev [Kle1]. Subsequently, the branching graph of Kleshchev was interpreted by Lascoux, Leclerc, and Thibon as the crystal graph of the basic representation of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_p)$. The observation [LLT] turned out to be a beginning of an exciting development which continues to this day, including a development of deep connections between (affine, cyclotomic or degenerate affine) Hecke algebras of type A at the ℓ th roots of unity and integrable $U_q(\widehat{\mathfrak{sl}}_\ell)$ -modules via categorification; see [Ari1, Br, BK, BKW, Gr, GV, Kle2] for related work.

1.2. Yokonuma-Hecke algebras were introduced by Yokonuma [Yo] as a centralizer algebra associated to the permutation representation of a finite Chevalley group G with respect to a maximal unipotent subgroup of G . The Yokonuma-Hecke algebra $Y_{r,n}(q)$ (of type A) is a quotient of the group algebra of the modular framed braid group $(\mathbb{Z}/r\mathbb{Z}) \wr B_n$, where B_n is the braid group on n strands (of type A). By the presentation given by Juyumaya and Kannan [Ju1, JuK], the Yokonuma-Hecke algebra $Y_{r,n}(q)$ can also be regraded as a deformation of the group algebra of the complex reflection group $G(r, 1, n)$, which is isomorphic to the wreath product $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$. It is well-known that there exists another deformation of the group algebra of $G(r, 1, n)$, namely the Ariki-Koike algebra [AK]. The Yokonuma-Hecke algebra $Y_{r,n}(q)$ is quite different from the Ariki-Koike algebra. For example, the Iwahori-Hecke algebra of type A is canonically a subalgebra of the Ariki-Koike algebra, whereas it is an obvious quotient of $Y_{r,n}(q)$, but not an obvious subalgebra of it.

Recently, by generalizing the approach of Okounkov-Vershik [OV] on the representation theory of \mathfrak{S}_n , Chlouveraki and Poulain d'Andecy [ChP1] introduced the notion of affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n}(q)$ and gave explicit formulas for all irreducible representations of $Y_{r,n}(q)$ over $\mathbb{C}(q)$, and obtained a semisimplicity criterion for it. In their

subsequent paper [ChP2], they studied the representation theory of the affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n}(q)$ and the cyclotomic Yokonuma-Hecke algebra $Y_{r,n}^d(q)$. In particular, they gave the classification of irreducible representations of $Y_{r,n}^d(q)$ in the generic semisimple case. In the past several years, the study of affine and cyclotomic Yokonuma-Hecke algebras has made substantial progress; see [ChP1, ChP2, ChS, C, ER, JP, Lu, Ro].

1.3. The second author and Wang [WW] have introduced the notion of wreath Hecke algebra associated to an arbitrary finite group G and developed its modular representation theory and modular branching rules. The wreath Hecke algebra (when G is the cyclic group of order r) can be regarded as a degeneration, when q tends to ± 1 , of the affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n}(q)$. Our goal of this paper is to develop the representation theory of the algebra $\widehat{Y}_{r,n}(q)$ by generalizing the approach of [WW]. The main results of this paper include the classification of the simple $\widehat{Y}_{r,n}(q)$ -modules as well as the classification of the simple modules of the cyclotomic Yokonuma-Hecke algebras over an algebraically closed field \mathbb{K} of characteristic p such that p does not divide r (which is required to make sure that the affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n}(q)$ is defined over \mathbb{K}). We also obtain the modular branching rule for $\widehat{Y}_{r,n}(q)$, and its interpretation via crystal graphs of quantum affine algebras.

1.4. We establish the PBW basis of the affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n}(q)$ and describe its center in Section 2.

Our study of the representation theory of the affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n}(q)$ is built on an equivalence between the category of finite dimensional $\widehat{Y}_{r,n}(q)$ -modules (over an algebraically closed field \mathbb{K} of characteristic p such that p does not divide r) and the module category of an algebra which is a direct sum of tensor products of various affine Hecke algebras $\mathcal{H}_{\mu_i}^{\mathbb{K}}$ of type A . This is achieved in Section 3.

In Section 4, we will give three applications of the above module category equivalence. First of all, we give the classification of finite dimensional $\widehat{Y}_{r,n}^{\mathbb{K}}$ -modules by a reduction to the known classification of simple modules for various algebras $\mathcal{H}_{\mu_i}^{\mathbb{K}}$. As a second application, we establish the modular branching rules for $\widehat{Y}_{r,n}^{\mathbb{K}}$. That is, we describe explicitly the socle of the restriction of a simple $\widehat{Y}_{r,n}^{\mathbb{K}}$ -module to a subalgebra $\widehat{Y}_{r,n-1,1}^{\mathbb{K}}$, and hence to the subalgebra $\widehat{Y}_{r,n-1}^{\mathbb{K}}$. Finally, we give a block decomposition in the category of finite dimensional $\widehat{Y}_{r,n}^{\mathbb{K}}$ -modules.

In Section 5, we establish an equivalence between the module category of finite dimensional modules of the cyclotomic Yokonuma-Hecke algebra $Y_{r,n}^{\lambda,\mathbb{K}}$ and the module category of an algebra which is a direct sum of tensor products of various cyclotomic Hecke algebras $\mathcal{H}_{\mu_i}^{\lambda,\mathbb{K}}$.

In Section 6, we present several applications of the above module category equivalence. First of all, we give the classification of finite dimensional $Y_{r,n}^{\lambda,\mathbb{K}}$ -modules by a reduction to the known classification of simple modules for various algebras $\mathcal{H}_{\mu_i}^{\lambda,\mathbb{K}}$. In particular, we establish the modular representation theory of the Yokonuma-Hecke algebra $Y_{r,n}^{\mathbb{K}}$. The second, we define an action of the affine Lie algebra, which is a direct sum of r -copies of \widehat{sl}_e , on the direct sum of the Grothendieck groups of $Y_{r,n}^{\lambda,\mathbb{K}}$ -modules for all $n \geq 0$, and

further show that the resulting representation is irreducible and integrable. The third, we establish the modular branching rules for $Y_{r,n}^{\lambda,\mathbb{K}}$. That is, we describe explicitly the socle of the restriction of a simple $Y_{r,n}^{\lambda,\mathbb{K}}$ -module to a subalgebra $Y_{r,n-1,1}^{\lambda,\mathbb{K}}$, and hence to the subalgebra $Y_{r,n-1}^{\lambda,\mathbb{K}}$. Furthermore, we show that the modular branching graph for $Y_{r,n}^{\lambda,\mathbb{K}}$ is isomorphic to the corresponding crystal graph of the simple $\widehat{sl}_e^{\oplus r}$ -module $L(\lambda)^{\otimes r}$. Finally, we give the classification of blocks for $Y_{r,n}^{\lambda,\mathbb{K}}$, which is reduced to the known classification for the cyclotomic Hecke algebra due to Lyle and Mathas [LM].

Throughout the paper: let $r, n \in \mathbb{Z}_{\geq 1}$, and let q be an indeterminate. Let $\mathcal{R} = \mathbb{Z}[\frac{1}{r}][q, q^{-1}]$, and let \mathbb{K} be an algebraically closed field of characteristic p such that p does not divide r . We remark that the assumption that p does not divide r is required so that the affine Yokonuma-Hecke algebras are defined over the field \mathbb{K} . We consider \mathbb{K} as an \mathcal{R} -algebra by mapping q to an invertible element $q \in \mathbb{K}^*$. If \mathcal{H} denotes an \mathcal{R} -algebra or an \mathcal{R} -module, then $\mathcal{H}^{\mathbb{K}} = \mathbb{K} \otimes_{\mathcal{R}} \mathcal{H}$ denotes the object obtained by base change to \mathbb{K} .

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2. THE DEFINITION AND PROPERTIES OF AFFINE YOKONUMA-HECKE ALGEBRAS

2.1. The definition of $\widehat{Y}_{r,n}(q)$.

Definition 2.1. The affine Yokonuma-Hecke algebra, denoted by $\widehat{Y}_{r,n}(q)$, is an \mathcal{R} -associative algebra generated by the elements $t_1, \dots, t_n, g_1, \dots, g_{n-1}, X_1^{\pm 1}$, in which the generators $t_1, \dots, t_n, g_1, \dots, g_{n-1}$ satisfy the following relations:

$$\begin{aligned} g_i g_j &= g_j g_i & \text{for all } i, j = 1, \dots, n-1 \text{ such that } |i-j| \geq 2, \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} & \text{for all } i = 1, \dots, n-2, \\ t_i t_j &= t_j t_i & \text{for all } i, j = 1, \dots, n, \\ g_i t_j &= t_{s_i(j)} g_i & \text{for all } i = 1, \dots, n-1 \text{ and } j = 1, \dots, n, \\ t_i^r &= 1 & \text{for all } i = 1, \dots, n, \\ g_i^2 &= 1 + (q - q^{-1}) e_i g_i & \text{for all } i = 1, \dots, n-1, \end{aligned} \tag{2.1}$$

where s_i is the transposition $(i, i+1)$ in the symmetric group \mathfrak{S}_n on n letters, and for each $1 \leq i \leq n-1$,

$$e_i := \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_{i+1}^{-s},$$

together with the following relations concerning the generators $X_1^{\pm 1}$:

$$\begin{aligned} X_1 X_1^{-1} &= X_1^{-1} X_1 = 1, \\ g_1 X_1 g_1 X_1 &= X_1 g_1 X_1 g_1, \\ g_i X_1 &= X_1 g_i & \text{for all } i = 2, \dots, n-1, \\ t_j X_1 &= X_1 t_j & \text{for all } j = 1, \dots, n, \end{aligned} \tag{2.2}$$

Remark 2.2. We recall that the Yokonuma-Hecke algebra $Y_{r,n}(q)$ of type A, defined by Yokonuma in [Yo], is the associative algebra over \mathcal{R} generated by elements t'_1, \dots, t'_n and g'_1, \dots, g'_{n-1} with the defining relations as in (2.1) with each g_i replaced by g'_i and each t_j

replaced by t'_j [Ju1, Ju2, JuK]. By [ChP2, (2.6)], the homomorphism $\iota : Y_{r,n}(q) \rightarrow \widehat{Y}_{r,n}(q)$, which is defined by

$$\iota(t'_j) = t_j \quad \text{for } 1 \leq j \leq n \quad \text{and} \quad \iota(g'_i) = g_i \quad \text{for } 1 \leq i \leq n-1, \quad (2.3)$$

is an injection. Meanwhile, By [ChP1, (3.6)], there exists a surjective algebra homomorphism $\pi : \widehat{Y}_{r,n}(q) \rightarrow Y_{r,n}(q)$ given by

$$\pi(t_j) = t'_j, \quad \pi(g_i) = g'_i, \quad \pi(X_1) = 1 \quad (2.4)$$

for $1 \leq j \leq n$ and $1 \leq i \leq n-1$.

By Remark 2.2, we can identify the Yokonuma-Hecke algebra $Y_{r,n}(q)$ with the subalgebra of $\widehat{Y}_{r,n}(q)$ generated by $t_1, \dots, t_n, g_1, \dots, g_{n-1}$. Moreover, let $G = \mathbb{Z}/r\mathbb{Z}$ and $T = G^n = (\mathbb{Z}/r\mathbb{Z})^n$. Then the group algebra of T over \mathcal{R} is isomorphic to the subalgebra of $\widehat{Y}_{r,n}(q)$ generated by t_1, \dots, t_n .

Note that the elements e_i are idempotents in $\widehat{Y}_{r,n}(q)$. The elements g_i are invertible, with the inverse given by

$$g_i^{-1} = g_i - (q - q^{-1})e_i \quad \text{for all } i = 1, \dots, n-1. \quad (2.5)$$

Let $w \in \mathfrak{S}_n$, and let $w = s_{i_1} \cdots s_{i_r}$ be a reduced expression of w . By Matsumoto's lemma, the element $g_w := g_{i_1} g_{i_2} \cdots g_{i_r}$ does not depend on the choice of the reduced expression of w .

Let $i, k \in \{1, 2, \dots, n\}$ and set

$$e_{i,k} := \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_k^{-s}. \quad (2.6)$$

Note that $e_{i,i} = 1$, $e_{i,k} = e_{k,i}$, and that $e_{i,i+1} = e_i$. It can be easily checked that the following holds:

$$e_{j,k} g_i = g_i e_{s_i(j), s_i(k)} \quad \text{for } i = 1, \dots, n-1 \text{ and } j, k = 1, \dots, n. \quad (2.7)$$

In particular, we have $e_i g_i = g_i e_i$ for all $i = 1, \dots, n-1$.

We define inductively elements X_2, \dots, X_n in $\widehat{Y}_{r,n}(q)$ by

$$X_{i+1} := g_i X_i g_i \quad \text{for } i = 1, \dots, n-1. \quad (2.8)$$

Then it is proved in [ChP1, Lemma 1] that we have, for any $1 \leq i \leq n-1$,

$$g_i X_j = X_j g_i \quad \text{for } j = 1, 2, \dots, n \text{ such that } j \neq i, i+1. \quad (2.9)$$

Moreover, by [ChP1, Proposition 1], we have that the elements $t_1, \dots, t_n, X_1, \dots, X_n$ form a commutative family, that is,

$$xy = yx \quad \text{for any } x, y \in \{t_1, \dots, t_n, X_1, \dots, X_n\}. \quad (2.10)$$

We shall often use the following identities (see [ChP2, Lemma 2.3]): for $1 \leq i \leq n-1$,

$$\begin{aligned} g_i X_i &= X_{i+1} g_i - (q - q^{-1}) e_i X_{i+1}, \\ g_i X_{i+1} &= X_i g_i + (q - q^{-1}) e_i X_{i+1}, \\ g_i X_i^{-1} &= X_{i+1}^{-1} g_i + (q - q^{-1}) e_i X_i^{-1}, \\ g_i X_{i+1}^{-1} &= X_i^{-1} g_i - (q - q^{-1}) e_i X_i^{-1}. \end{aligned} \quad (2.11)$$

2.2. The center of $\widehat{Y}_{r,n}^{\mathbb{K}}$. From now on, we always consider the specializations over \mathbb{K} of various algebras:

$$Y_{r,n}^{\mathbb{K}} = \mathbb{K} \otimes_{\mathcal{R}} Y_{r,n}(q), \quad \widehat{Y}_{r,n}^{\mathbb{K}} = \mathbb{K} \otimes_{\mathcal{R}} \widehat{Y}_{r,n}(q).$$

Recall that $G = \mathbb{Z}/r\mathbb{Z}$ and $T = G^n$. Observe that the symmetric group \mathfrak{S}_n acts on T by permutations: ${}^w h := (h_{w^{-1}(1)}, \dots, h_{w^{-1}(n)})$ for any $h = (h_1, \dots, h_n) \in T$ and $w \in \mathfrak{S}_n$. Let $P_n^{\mathbb{K}} = \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ be the algebra of Laurent polynomials in X_1, \dots, X_n , which is regarded as a subalgebra of $\widehat{Y}_{r,n}^{\mathbb{K}}$. For each $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, set $X^\alpha = X_1^{\alpha_1} \dots X_n^{\alpha_n}$. The symmetric group \mathfrak{S}_n acts as automorphisms on $P_n^{\mathbb{K}}$ by permutations. Let us denote this action by $f \mapsto {}^w f$ for $w \in \mathfrak{S}_n$ and $f \in P_n^{\mathbb{K}}$. Then we have ${}^w(X^\alpha) = X^{w\alpha}$, where $w\alpha = (\alpha_{w^{-1}(1)}, \dots, \alpha_{w^{-1}(n)})$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and $w \in \mathfrak{S}_n$.

By making use of the identities (2.11) and by induction on the degree of the polynomials, we can easily get the following lemma.

Lemma 2.3. *For any $f \in P_n^{\mathbb{K}}$ and $1 \leq i \leq n-1$, The following holds:*

$$g_i f - {}^{s_i} f g_i = (q - q^{-1}) e_i \frac{f - {}^{s_i} f}{1 - X_i X_{i+1}^{-1}}. \quad (2.12)$$

The next lemma easily follows from Lemma 2.3.

Lemma 2.4. *Let $w \in \mathfrak{S}_n$, $t \in T$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$. Denote the Bruhat order on \mathfrak{S}_n by \leq . Then in $\widehat{Y}_{r,n}^{\mathbb{K}}$, we have*

$$g_w t X^\alpha = ({}^w t) X^{w\alpha} g_w + \sum_{u < w} t_u f_u g_u, \quad t X^\alpha g_w = g_w ({}^{w^{-1}} t) X^{w^{-1}\alpha} + \sum_{u < w} g_u t'_u f'_u$$

for some $f_u, f'_u \in P_n^{\mathbb{K}}$ and $t_u, t'_u \in \mathbb{K}T$.

The following theorem gives the PBW basis for the affine Yokonuma-Hecke algebra $\widehat{Y}_{r,n}^{\mathbb{K}}$ (see also [ChP2, Theorem 4.4]).

Theorem 2.5. *Let $\mathcal{H}_n^{\mathbb{K}}$ be the \mathbb{K} -vector space spanned by the elements g_w for $w \in \mathfrak{S}_n$. Then we have an isomorphism of vector spaces*

$$P_n^{\mathbb{K}} \otimes \mathbb{K}T \otimes \mathcal{H}_n^{\mathbb{K}} \longrightarrow \widehat{Y}_{r,n}^{\mathbb{K}}.$$

That is, the elements $\{X^\alpha t g_w \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n, t \in T, w \in \mathfrak{S}_n\}$ form a \mathbb{K} -basis of $\widehat{Y}_{r,n}^{\mathbb{K}}$, which is called the PBW basis.

Proof. It follows from Lemma 2.4 that $\widehat{Y}_{r,n}^{\mathbb{K}}$ is spanned by the elements $X^\alpha t g_w$ for $\alpha \in \mathbb{Z}^n$, $t \in T$, and $w \in \mathfrak{S}_n$. Since the set $\{h \otimes Y^\alpha \mid h \in T, \alpha \in \mathbb{Z}^n\}$ forms a \mathbb{K} -basis for the vector space $\mathbb{K}T \otimes_{\mathbb{K}} \mathbb{K}[Y_1^{\pm 1}, \dots, Y_n^{\pm 1}]$, we can verify by a direct calculation that $\mathbb{K}T \otimes_{\mathbb{K}} \mathbb{K}[Y_1^{\pm 1}, \dots, Y_n^{\pm 1}]$ is a $\widehat{Y}_{r,n}^{\mathbb{K}}$ -module, which is defined by

$$\begin{aligned} X_i^{\pm 1} \circ (h \otimes Y^\alpha) &= h \otimes Y_i^{\pm 1} Y^\alpha & \text{for } 1 \leq i \leq n, \\ t \circ (h \otimes Y^\alpha) &= t h \otimes Y^\alpha & \text{for } t \in T, \\ g_j \circ (h \otimes Y^\alpha) &= {}^{s_j} h \otimes Y^{s_j \alpha} + (q - q^{-1}) ({}^{s_j} h) e_j \otimes \frac{Y^\alpha - Y^{s_j \alpha}}{1 - Y_j Y_{j+1}^{-1}} & \text{for } 1 \leq j \leq n-1. \end{aligned}$$

In order to show that the elements $X^\alpha t g_w$ are linearly independent, it suffices to prove that they act as linearly independent linear operators on $\mathbb{K}T \otimes_{\mathbb{K}} \mathbb{K}[Y_1^{\pm 1}, \dots, Y_n^{\pm 1}]$. But this is clear if we consider the action on an element of the form $Y_1^N Y_2^{2N} \dots Y_n^{nN}$ for $N \gg 0$. \square

Let $P_n^{\mathbb{K}}(T)$ be the subalgebra of $\widehat{Y}_{r,n}^{\mathbb{K}}$ generated by t_1, \dots, t_n and $X_1^{\pm 1}, \dots, X_n^{\pm 1}$. Then we have

$$P_n^{\mathbb{K}}(T) \cong P_n^{\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{K}T.$$

Lemma 2.6. *The center of $\widehat{Y}_{r,n}^{\mathbb{K}}$ is contained in the subalgebra $P_n^{\mathbb{K}}(T)$.*

Proof. Take a central element $z = \sum_{w \in \mathfrak{S}_n} z_w g_w \in \widehat{Y}_{r,n}^{\mathbb{K}}$, where $z_w = \sum d_{t,\alpha} X^\alpha t \in P_n^{\mathbb{K}}(T)$. Let τ be maximal with respect to the Bruhat order such that $z_\tau \neq 0$. Assume that $\tau \neq 1$. Then there exists some $i \in \{1, 2, \dots, n\}$ with $\tau(i) \neq i$. By Lemma 2.4, we have

$$X_i z - z X_i = z_\tau (X_i - X_{\tau(i)}) g_\tau + \sum_{u < \tau} a_{t', \beta, u} X^\beta t' g_u.$$

By Theorem 2.5, we must have $z_\tau = 0$, which is a contradiction. Hence we must have $\tau = 1$ and $z \in P_n^{\mathbb{K}}(T)$. \square

Let $P_n^{\mathbb{K}}(T)^{\mathfrak{S}_n} = \{\sum d_{\alpha, \beta} X^\alpha t^\beta \in P_n^{\mathbb{K}}(T) \mid \sum d_{\alpha, \beta} X^\alpha t^\beta = \sum d_{\alpha, \beta} X^{w\alpha} t^{w\beta} \text{ for any } w \in \mathfrak{S}_n\}$.

Theorem 2.7. *The center of $\widehat{Y}_{r,n}^{\mathbb{K}}$ consists of elements of the form $z = \sum d_{\alpha, \beta} X^\alpha t^\beta$ satisfying $d_{w\alpha, w\beta} = d_{\alpha, \beta}$ for any $w \in \mathfrak{S}_n$ and $\alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}_r^n$. Thus, $Z(\widehat{Y}_{r,n}^{\mathbb{K}}) = P_n^{\mathbb{K}}(T)^{\mathfrak{S}_n}$.*

Proof. Suppose that

$$z = \sum_{\alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}_r^n} d_{\alpha, \beta} X^\alpha t^\beta \in Z(\widehat{Y}_{r,n}^{\mathbb{K}}).$$

Then we have, for each $1 \leq k \leq n-1$, $g_k z = z g_k$, that is, $g_k \cdot \sum d_{\alpha, \beta} X^\alpha t^\beta = \sum d_{\alpha, \beta} X^\alpha t^\beta g_k$. Thus, by (2.4) we have

$$\sum d_{\alpha, \beta} X^{s_k \alpha} t^{s_k \beta} g_k + (q - q^{-1}) \sum_{\alpha, \beta} d_{\alpha, \beta} e_k \frac{X^\alpha - X^{s_k \alpha}}{1 - X_k X_{k+1}^{-1}} t^\beta = \sum_{\alpha, \beta} d_{\alpha, \beta} X^\alpha t^\beta g_k.$$

By Theorem 2.5, we must have

$$\sum_{\alpha, \beta} d_{\alpha, \beta} X^{s_k \alpha} t^{s_k \beta} = \sum_{\alpha, \beta} d_{\alpha, \beta} X^\alpha t^\beta \quad \text{for any } 1 \leq k \leq n-1, \quad (2.13)$$

$$\sum_{\alpha, \beta} d_{\alpha, \beta} e_k \frac{X^\alpha - X^{s_k \alpha}}{1 - X_k X_{k+1}^{-1}} t^\beta = 0 \quad \text{for any } 1 \leq k \leq n-1. \quad (2.14)$$

We claim that (2.13) implies (2.14). In fact, for each $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_r^n$ and $1 \leq k \leq n-1$, we have

$$\begin{aligned}
e_k t^\beta &= \left(\frac{1}{r} \sum_{s=0}^{r-1} t_k^s t_{k+1}^{-s} \right) t_1^{\beta_1} \dots t_n^{\beta_n} = \frac{1}{r} \sum_{s=0}^{r-1} t_1^{\beta_1} \dots t_{k-1}^{\beta_{k-1}} t_k^{\beta_k+s} t_{k+1}^{\beta_{k+1}-s} t_{k+2}^{\beta_{k+2}} \dots t_n^{\beta_n} \\
&= \frac{1}{r} \sum_{s=0}^{r-1} t_1^{\beta_1} \dots t_{k-1}^{\beta_{k-1}} t_k^{\beta_k} t_{k+1}^{\beta_{k+1}} (t_k^{\beta_k-\beta_{k+1}+s} t_{k+1}^{\beta_{k+1}-\beta_k-s}) t_{k+2}^{\beta_{k+2}} \dots t_n^{\beta_n} \\
&= t_1^{\beta_1} \dots t_{k-1}^{\beta_{k-1}} t_k^{\beta_k} t_{k+1}^{\beta_{k+1}} t_{k+2}^{\beta_{k+2}} \dots t_n^{\beta_n} \left(\frac{1}{r} \sum_{s=0}^{r-1} t_k^{\beta_k-\beta_{k+1}+s} t_{k+1}^{\beta_{k+1}-\beta_k-s} \right) \\
&= t^{s_k \beta} \left(\frac{1}{r} \sum_{s'=0}^{r-1} t_k^{s'} t_{k+1}^{-s'} \right) = t^{s_k \beta} e_k, \tag{2.15}
\end{aligned}$$

where $s' = \beta_k - \beta_{k+1} + s \pmod{r}$. Then we have

$$\begin{aligned}
\sum d_{\alpha, \beta} e_k X^\alpha t^\beta &= e_k z = z e_k \text{ since } z \text{ is central,} \\
&= \sum d_{\alpha, \beta} X^{s_k \alpha} t^{s_k \beta} e_k \text{ by (2.13),} \\
&= \sum d_{\alpha, \beta} X^{s_k \alpha} e_k t^\beta \text{ by (2.15),} \\
&= \sum d_{\alpha, \beta} e_k X^{s_k \alpha} t^\beta.
\end{aligned}$$

This is an invariant of (2.14) with the denominator cleared.

Note now that (2.13) holds if and only if $d_{\alpha, \beta} = d_{s_k \alpha, s_k \beta}$ for $\alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}_r^n$ and $1 \leq k \leq n-1$, and hence $d_{w\alpha, w\beta} = d_{\alpha, \beta}$ for any $w \in \mathfrak{S}_n$.

Reversing the above arguments, an element $z \in \widehat{Y}_{r,n}^{\mathbb{K}}$ of the form $z = \sum d_{\alpha, \beta} X^\alpha t^\beta$ satisfying $d_{w\alpha, w\beta} = d_{\alpha, \beta}$ for any $w \in \mathfrak{S}_n$ is indeed central. \square

Corollary 2.8. *If M is an irreducible $\widehat{Y}_{r,n}^{\mathbb{K}}$ -module, then M is finite dimensional.*

Proof. It is known that $P_n^{\mathbb{K}}$ is a free $\mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{\mathfrak{S}_n}$ -module of finite rank $n!$, and $\mathbb{K}T$ is a free $(\mathbb{K}T)^{\mathfrak{S}_n}$ -module of finite rank. Hence by Theorem 2.7 we observe that $\widehat{Y}_{r,n}^{\mathbb{K}}$ is a free module over its center $Z(\widehat{Y}_{r,n}^{\mathbb{K}})$ of finite rank. Dixmier's version of Schur's lemma implies that the center of $\widehat{Y}_{r,n}^{\mathbb{K}}$ acts by scalars on absolutely irreducible modules, which implies that M is an irreducible module for a finite dimensional algebra, and hence M is finite dimensional. \square

Remark 2.9. Recently, Chlouveraki and Sécherre [ChS, Theorem 4.3] proved that the affine Yokonuma-Hecke algebra is a particular case of the pro- p -Iwahori-Hecke algebra defined by Vignéras in [Vi1]. In [Vi2, Theorem 1.3] Vignéras described the center of the pro- p -Iwahori-Hecke algebra over any commutative ring R . Thus, our Theorem 2.7 can be regarded as a particular case of Vignéras' results.

3. AN EQUIVALENCE OF MODULE CATEGORIES

In this section, we establish an explicit equivalence between the category $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$ of finite dimensional $\widehat{Y}_{r,n}^{\mathbb{K}}$ -modules and the category $\widehat{\mathcal{H}}_{r,n}^{\mathbb{K}}\text{-mod}$ of finite dimensional $\widehat{\mathcal{H}}_{r,n}^{\mathbb{K}}$ -modules, where $\widehat{\mathcal{H}}_{r,n}^{\mathbb{K}}$ is a direct sum of tensor products for various affine Hecke algebras $\widehat{\mathcal{H}}_{\mu_i}^{\mathbb{K}}$ of type A . This category equivalence plays a crucial role throughout the rest of this paper.

3.1. The structure of $\widehat{Y}_{r,n}^{\mathbb{K}}$ -modules. Let $\{V_1, \dots, V_a\}$ be a complete set of pairwise non-isomorphic finite dimensional simple $\mathbb{K}G$ -modules. Since \mathbb{K} is an algebraically closed field of characteristic p such that p does not divide r and G is the cyclic group $\mathbb{Z}/r\mathbb{Z}$, we have $a = r$ and $\dim V_k = 1$ for each $1 \leq k \leq r$. Using this fact, we can easily get the next lemma, which can be regarded as a particular case of [WW, Lemma 3.1]. Recall that $e_i = \frac{1}{r} \sum_{s=0}^{r-1} t_i^s t_{i+1}^{-s}$ for $1 \leq i \leq n-1$.

Lemma 3.1. (1) $e_1 = 0$, when acting on a simple $\mathbb{K}G^2$ -module $V_k \otimes V_l$ for $1 \leq k \neq l \leq r$.
 (2) $e_1 = \text{id}$, when acting on the $\mathbb{K}G^2$ -module $V_k^{\otimes 2}$ for $1 \leq k \leq r$.

Since $\{V_{i_1} \otimes \dots \otimes V_{i_n} \mid 1 \leq i_1, \dots, i_n \leq r\}$ forms a complete set of pairwise non-isomorphic simple $\mathbb{K}T$ -modules, by Lemma 3.1, we immediately get that on $V_{i_1} \otimes \dots \otimes V_{i_n}$, e_k acts as the identity if $i_k = i_{k+1}$; otherwise, e_k acts as zero.

Set $\mathbb{I} := \{q^i \mid i \in \mathbb{Z}\}$. Let e denote the number of elements in \mathbb{I} . Then $e \in \mathbb{N} \cup \{\infty\}$, and e is the order of $q \in \mathbb{K}^*$.

Given an algebra S , we denote by $S\text{-mod}$ the category of finite dimensional left S -modules. Since \mathbb{K} is an algebraically closed field of characteristic p such that p does not divide r , every module M in $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$ is semisimple when restricted to the subalgebra $\mathbb{K}T$.

Let $\mathcal{C}_r(n)$ be the set of r -compositions of n , that is, the set of r -tuples of non-negative integers $\mu = (\mu_1, \dots, \mu_r)$ such that $\sum_{1 \leq a \leq r} \mu_a = n$. For each $\mu \in \mathcal{C}_r(n)$, let

$$V(\mu) = V_1^{\otimes \mu_1} \otimes \dots \otimes V_r^{\otimes \mu_r}$$

be the corresponding simple $\mathbb{K}T$ -module. Let $\mathfrak{S}_\mu := \mathfrak{S}_{\mu_1} \times \dots \times \mathfrak{S}_{\mu_r}$ be the corresponding Young subgroup of \mathfrak{S}_n and denote by $\mathcal{O}(\mu)$ a complete set of left coset representatives of \mathfrak{S}_μ in \mathfrak{S}_n . For each $\mu \in \mathcal{C}_r(n)$, we define $\widehat{Y}_{r,\mu}^{\mathbb{K}}$ to be the subalgebra of $\widehat{Y}_{r,n}^{\mathbb{K}}$ generated by t_1, \dots, t_n , $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ and g_w for $w \in \mathfrak{S}_\mu$. Then by Definition 2.1 we have

$$\widehat{Y}_{r,\mu}^{\mathbb{K}} \cong \widehat{Y}_{r,\mu_1}^{\mathbb{K}} \otimes \dots \otimes \widehat{Y}_{r,\mu_r}^{\mathbb{K}}.$$

And every module N in $\widehat{Y}_{r,\mu}^{\mathbb{K}}\text{-mod}$ is semisimple when restricted to $\mathbb{K}T$.

Given an $M \in \widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$, we define $I_\mu M$ to be the isotypical subspace of $V(\mu)$ in M , that is, the sum of all simple $\mathbb{K}T$ -submodules of M isomorphic to $V(\mu)$. We define M_μ by

$$M_\mu := \sum_{w \in \mathfrak{S}_n} g_w(I_\mu M).$$

Lemma 3.2. Let $\mu \in \mathcal{C}_r(n)$ and $M \in \widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$. Then, $I_\mu M$ is a $\widehat{Y}_{r,\mu}^{\mathbb{K}}$ -submodule and M_μ is a $\widehat{Y}_{r,n}^{\mathbb{K}}$ -submodule of M . Moreover, $M_\mu \cong \text{Ind}_{\widehat{Y}_{r,\mu}^{\mathbb{K}}}^{\widehat{Y}_{r,n}^{\mathbb{K}}}(I_\mu M)$.

Proof. Since $X_i^{\pm 1}$ commutes with $\mathbb{K}T$ for each $1 \leq i \leq n$, then each $X_i^{\pm 1}$ ($1 \leq i \leq n$) maps a simple $\mathbb{K}T$ -submodule of M to an isomorphic copy. Hence, $I_\mu M$ is invariant under the action of the subalgebra $P_n^{\mathbb{K}}$. Since g_w , for each $w \in \mathfrak{S}_\mu$, maps a simple $\mathbb{K}T$ -submodules of M isomorphic to $V(\mu)$ to another isomorphic one, $I_\mu M$ is invariant under the action of g_w for all $w \in \mathfrak{S}_\mu$. Hence, $I_\mu M$ is a $\widehat{Y}_{r,\mu}^{\mathbb{K}}$ -submodule, since $\widehat{Y}_{r,\mu}^{\mathbb{K}}$ is generated by $P_n^{\mathbb{K}}$, $\mathbb{K}T$, and g_w ($w \in \mathfrak{S}_\mu$).

It follows from the definition that M_μ is a $\widehat{Y}_{r,n}^{\mathbb{K}}$ -submodule of M .

By Frobenius reciprocity, we have a nonzero $\widehat{Y}_{r,n}^{\mathbb{K}}$ -homomorphism

$$\phi : \text{Ind}_{\widehat{Y}_{r,\mu}^{\mathbb{K}}}^{\widehat{Y}_{r,n}^{\mathbb{K}}}(I_\mu M) \rightarrow M_\mu.$$

Observe that

$$M_\mu = \sum_{w \in \mathfrak{S}_n} g_w(I_\mu M) = \sum_{\tau \in \mathcal{O}(\mu)} g_\tau(I_\mu M).$$

Hence, ϕ is surjective, and then an isomorphism by counting dimensions. \square

Lemma 3.3. *We have the following decomposition in $\widehat{Y}_{r,n}^{\mathbb{K}}$ -mod:*

$$M = \bigoplus_{\mu \in \mathcal{C}_r(n)} M_\mu.$$

Proof. Let $M \in \widehat{Y}_{r,n}^{\mathbb{K}}$ -mod. Then M is semisimple as a $\mathbb{K}T$ -module. Observe that M_μ is the direct sum of those isotypical components of simple $\mathbb{K}T$ -modules which contain exactly μ_i tensor factors isomorphic to V_i for all $1 \leq i \leq r$. Now the lemma follows. \square

3.2. An equivalence of categories. For each $n \in \mathbb{N}$, the extended affine Hecke algebra $\widehat{\mathcal{H}}_n$ of type A is a $\mathbb{Z}[q, q^{-1}]$ -algebra generated by elements $T_i, Y_j^{\pm 1}$, where $1 \leq i \leq n-1$ and $1 \leq j \leq n$, subject to the following relations:

- (1) $(T_i - q)(T_i + q^{-1}) = 0$, $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $i = 1, 2, \dots, n-1$;
- (2) $T_i T_j = T_j T_i$ for $|i - j| \geq 2$;
- (3) $Y_i Y_i^{-1} = Y_i^{-1} Y_i = 1$, $Y_i Y_j = Y_j Y_i$ for all i, j ;
- (4) $T_i Y_i T_i = Y_{i+1}$ for $i = 1, 2, \dots, n-1$, $T_i Y_j = Y_j T_i$ for $j \neq i, i+1$.

Let $w \in \mathfrak{S}_n$, and let $w = s_{i_1} \cdots s_{i_k}$ be a reduced expression of w . The element $T_w := T_{i_1} T_{i_2} \cdots T_{i_k}$ does not depend on the choice of the reduced expression of w . Note that $\widehat{\mathcal{H}}_n^{\mathbb{K}} = \mathbb{K} \otimes_{\mathbb{Z}[q, q^{-1}]} \widehat{\mathcal{H}}_n$.

We define the following algebra:

$$\widehat{\mathcal{H}}_{r,n}^{\mathbb{K}} := \bigoplus_{\mu \in \mathcal{C}_r(n)} \widehat{\mathcal{H}}_{r,\mu}^{\mathbb{K}}, \text{ where } \widehat{\mathcal{H}}_{r,\mu}^{\mathbb{K}} = \widehat{\mathcal{H}}_{\mu_1}^{\mathbb{K}} \otimes \cdots \otimes \widehat{\mathcal{H}}_{\mu_r}^{\mathbb{K}}.$$

Recall that $\{V_1, \dots, V_r\}$ is a complete set of pairwise non-isomorphic finite dimensional simple $\mathbb{K}G$ -modules and moreover $\dim V_i = 1$ for $1 \leq i \leq r$. So we can write $V_i = \mathbb{K}v_i$ for $1 \leq i \leq r$. For each $\mu \in \mathcal{C}_r(n)$, set $v_\mu = v_1^{\otimes \mu_1} \otimes \cdots \otimes v_r^{\otimes \mu_r} \in V(\mu)$. And then $V(\mu) = \mathbb{K}v_\mu$.

Proposition 3.4. *Let $\mu \in \mathcal{C}_r(n)$ and $N \in \widehat{Y}_{r,\mu}^{\mathbb{K}}\text{-mod}$. Then $\text{Hom}_{\mathbb{K}T}(V(\mu), N)$ is an $\widehat{\mathcal{H}}_{r,\mu}^{\mathbb{K}}$ -module with the action given by*

$$(T_w \diamond \phi)(v_\mu) = g_w \phi(v_\mu),$$

$$(Y_k^{\pm 1} \diamond \phi)(v_\mu) = X_k^{\pm 1} \phi(v_\mu)$$

for $w \in \mathfrak{S}_\mu$, $1 \leq k \leq n$, and $\phi \in \text{Hom}_{\mathbb{K}T}(V(\mu), N)$. Thus, $\text{Hom}_{\mathbb{K}T}(V(\mu), -)$ is a functor from $\widehat{Y}_{r,\mu}^{\mathbb{K}}\text{-mod}$ to $\widehat{\mathcal{H}}_{r,\mu}^{\mathbb{K}}\text{-mod}$.

Proof. Let us first show that $T_w \diamond \phi$ is a $\mathbb{K}T$ -homomorphism. It suffices to consider each $T_i \diamond \phi$ for $i \in I_\mu := \{1, 2, \dots, n-1\} \setminus \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \dots + \mu_{r-1}\}$. Observe that we have, for each $1 \leq j \leq n$,

$$\begin{aligned} (T_i \diamond \phi)(t_j(v_\mu)) &= (T_i \diamond \phi)(t_{s_i(j)}(v_\mu)) \\ &= g_i \phi(t_{s_i(j)}(v_\mu)) \\ &= g_i t_{s_i(j)} \phi(v_\mu) \\ &= t_j(T_i \diamond \phi)(v_\mu). \end{aligned}$$

The fact that $Y_k^{\pm 1} \diamond \phi$ is a $\mathbb{K}T$ -homomorphism can be proved similarly.

Using the fact that e_k , for each $k \in I_\mu$, acts on $V(\mu)$ as the identity, it is easy to verify the relations for the $\widehat{\mathcal{H}}_{r,\mu}^{\mathbb{K}}$ -module structure on $\text{Hom}_{\mathbb{K}T}(V(\mu), N)$. We will skip the details. \square

Proposition 3.5. *Let M be an $\widehat{\mathcal{H}}_{r,\mu}^{\mathbb{K}}$ -module. Then $V(\mu) \otimes M$ is a $\widehat{Y}_{r,\mu}^{\mathbb{K}}$ -module via*

$$\begin{aligned} t_k * (v_\mu \otimes z) &= t_k(v_\mu) \otimes z, \\ g_w * (v_\mu \otimes z) &= v_\mu \otimes T_w z, \\ X_k^{\pm 1} * (v_\mu \otimes z) &= v_\mu \otimes Y_k^{\pm 1} z \end{aligned}$$

for $1 \leq k \leq n$, $w \in \mathfrak{S}_\mu$ and $z \in M$. There exists an isomorphism of $\widehat{\mathcal{H}}_{r,\mu}^{\mathbb{K}}$ -modules $\Phi : M \rightarrow \text{Hom}_{\mathbb{K}T}(V(\mu), V(\mu) \otimes M)$ given by $\Phi(z)(v) = v \otimes z$. Moreover, $V(\mu) \otimes M$ is a simple $\widehat{Y}_{r,\mu}^{\mathbb{K}}$ -module if and only if M is a simple $\widehat{\mathcal{H}}_{r,\mu}^{\mathbb{K}}$ -module.

Proof. It is straightforward to verify that $V(\mu) \otimes M$ is a $\widehat{Y}_{r,\mu}^{\mathbb{K}}$ -module as given above.

It is easy to see that Φ is a well-defined injective $\widehat{\mathcal{H}}_{r,\mu}^{\mathbb{K}}$ -homomorphism. However, observe that as a $\mathbb{K}T$ -module, $V(\mu) \otimes M$ is isomorphic to a direct sum of copies of $V(\mu)$. Thus, Φ is an isomorphism by comparing dimensions of these two modules.

Suppose that $V(\mu) \otimes M$ is a simple $\widehat{Y}_{r,\mu}^{\mathbb{K}}$ -module and E is a nonzero $\widehat{\mathcal{H}}_{r,\mu}^{\mathbb{K}}$ -submodule of M . Then $V(\mu) \otimes E$ is a nonzero $\widehat{Y}_{r,\mu}^{\mathbb{K}}$ -submodule of $V(\mu) \otimes M$, which implies $E = M$. Conversely, suppose that M is a simple $\widehat{\mathcal{H}}_{r,\mu}^{\mathbb{K}}$ -module and P is a nonzero $\widehat{Y}_{r,\mu}^{\mathbb{K}}$ -submodule of $V(\mu) \otimes M$. By Proposition 3.4, $\text{Hom}_{\mathbb{K}T}(V(\mu), P)$ is a nonzero $\widehat{\mathcal{H}}_{r,\mu}^{\mathbb{K}}$ -submodule of $\text{Hom}_{\mathbb{K}T}(V(\mu), V(\mu) \otimes M) \cong M$, which is simple. Hence, $\text{Hom}_{\mathbb{K}T}(V(\mu), P) \cong M$. Since P , as a $\mathbb{K}T$ -module, is isomorphic to a direct sum of copies of $V(\mu)$, we must have $P = V(\mu) \otimes M$ by a dimension comparison. \square

Proposition 3.6. *Let $N \in \widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$. Then we have*

$$\begin{aligned} \Psi : V(\mu) \otimes \text{Hom}_{\mathbb{K}T}(V(\mu), I_\mu N) &\longrightarrow I_\mu N, \\ v_\mu \otimes \psi &\mapsto \psi(v_\mu) \end{aligned}$$

defines an isomorphism of $\widehat{Y}_{r,\mu}^{\mathbb{K}}$ -modules.

Proof. By Lemma 3.2, $I_\mu N$ is a $\widehat{Y}_{r,\mu}^{\mathbb{K}}$ -module. It follows from Propositions 3.4 and 3.5 that $V(\mu) \otimes \text{Hom}_{\mathbb{K}T}(V(\mu), I_\mu N)$ is a $\widehat{Y}_{r,\mu}^{\mathbb{K}}$ -module.

It can be easily checked that Ψ is a $\widehat{Y}_{r,\mu}^{\mathbb{K}}$ -homomorphism. Since as a $\mathbb{K}T$ -module $I_\mu N$ is isomorphic to a direct sum of copies of $V(\mu)$, Ψ is surjective, and hence an isomorphism by a dimension comparison. \square

We now give one of the main results of this paper.

Theorem 3.7. *The functor $\mathcal{F} : \widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod} \rightarrow \widehat{\mathcal{H}}_{r,n}^{\mathbb{K}}\text{-mod}$ defined by*

$$\mathcal{F}(N) = \bigoplus_{\mu \in \mathcal{C}_r(n)} \text{Hom}_{\mathbb{K}T}(V(\mu), I_\mu N)$$

is an equivalence of categories with the inverse $\mathcal{G} : \widehat{\mathcal{H}}_{r,n}^{\mathbb{K}}\text{-mod} \rightarrow \widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$ given by

$$\mathcal{G}(\bigoplus_{\mu \in \mathcal{C}_r(n)} P_\mu) = \bigoplus_{\mu \in \mathcal{C}_r(n)} \text{Ind}_{\widehat{Y}_{r,\mu}^{\mathbb{K}}}^{\widehat{Y}_{r,n}^{\mathbb{K}}}(V(\mu) \otimes P_\mu).$$

Proof. Note that the map Φ in Proposition 3.5 is natural in M and Ψ in Proposition 3.6 is natural in N . One can easily check that $\mathcal{F}\mathcal{G} \cong \text{id}$ and $\mathcal{G}\mathcal{F} \cong \text{id}$ by using Lemmas 3.2 and 3.3, and Propositions 3.4-3.6. \square

4. CLASSIFICATION OF SIMPLE MODULES AND MODULAR BRANCHING RULES

In this section, we will present three applications of the equivalence of module categories established in Section 3. We shall classify all finite dimensional simple $\widehat{Y}_{r,n}^{\mathbb{K}}$ -modules, and establish the modular branching rule for $\widehat{Y}_{r,n}^{\mathbb{K}}$ which provides a description of the socle of the restriction to $\widehat{Y}_{r,n-1,1}^{\mathbb{K}}$ of a simple $\widehat{Y}_{r,n}^{\mathbb{K}}$ -module. We also give a block decomposition of $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$.

4.1. The simple $\widehat{Y}_{r,n}^{\mathbb{K}}$ -modules.

Theorem 4.1. *Each simple $\widehat{Y}_{r,n}^{\mathbb{K}}$ -module is isomorphic to a module of the form*

$$S_\mu(L.) := \text{Ind}_{\widehat{Y}_{r,\mu}^{\mathbb{K}}}^{\widehat{Y}_{r,n}^{\mathbb{K}}}((V_1^{\otimes \mu_1} \otimes L_1) \otimes \cdots \otimes (V_r^{\otimes \mu_r} \otimes L_r)),$$

where $\mu = (\mu_1, \dots, \mu_r) \in \mathcal{C}_r(n)$, and L_k ($1 \leq k \leq r$) is a simple $\widehat{\mathcal{H}}_{\mu_k}^{\mathbb{K}}$ -module. Moreover, the above modules $S_\mu(L.)$, for varied $\mu \in \mathcal{C}_r(n)$ and L_k ($1 \leq k \leq r$), form a complete set of pairwise non-isomorphic simple $\widehat{Y}_{r,n}^{\mathbb{K}}$ -modules.

Proof. It follows from the category equivalence established in Theorem 3.7. \square

Remark 4.2. It is known that Ariki and Mathas have given the classification of the simple modules of an affine Hecke algebra of type A over an arbitrary field in terms of aperiodic multisegments. In particular, the non-isomorphic simple $\widehat{\mathcal{H}}_n^{\mathbb{K}}$ -modules are indexed by the set $\mathcal{M}_e^n(\mathbb{K})$ (see [AM, Theorem B(i)] for the details), where e is the order of q in \mathbb{K} . Combining this with Theorem 4.1, we obtain that the simple $\widehat{Y}_{r,n}^{\mathbb{K}}$ -modules are indexed by the set

$$\mathcal{A} = \{(\mu, \psi_1, \dots, \psi_r) \mid \mu = (\mu_1, \dots, \mu_r) \in \mathcal{C}_r(n), \psi_i \in \mathcal{M}_e^{\mu_i}(\mathbb{K}), 1 \leq i \leq r\}.$$

4.2. Modular branching rules for $\widehat{Y}_{r,n}^{\mathbb{K}}$. For $a \in \mathbb{K}^*$ and $M \in \widehat{\mathcal{H}}_n^{\mathbb{K}}\text{-mod}$, let $\Delta_a(M)$ be the generalized a -eigenspace of Y_n in $\text{Res}_{\widehat{\mathcal{H}}_{n-1,1}^{\mathbb{K}}}^{\widehat{\mathcal{H}}_n^{\mathbb{K}}} M$, where $\widehat{\mathcal{H}}_{n-1,1}^{\mathbb{K}} = \widehat{\mathcal{H}}_{n-1}^{\mathbb{K}} \otimes \widehat{\mathcal{H}}_1^{\mathbb{K}}$. Since $Y_n - a$ is central in the subalgebra $\widehat{\mathcal{H}}_{n-1,1}^{\mathbb{K}}$ of $\widehat{\mathcal{H}}_n^{\mathbb{K}}$, $\Delta_a(M)$ is an $\widehat{\mathcal{H}}_{n-1,1}^{\mathbb{K}}$ -submodule of $\text{Res}_{\widehat{\mathcal{H}}_{n-1,1}^{\mathbb{K}}}^{\widehat{\mathcal{H}}_n^{\mathbb{K}}} M$. Define

$$e_a M := \text{Res}_{\widehat{\mathcal{H}}_{n-1}^{\mathbb{K}}}^{\widehat{\mathcal{H}}_{n-1,1}^{\mathbb{K}}} \Delta_a(M).$$

Then we have

$$\text{Res}_{\widehat{\mathcal{H}}_{n-1}^{\mathbb{K}}}^{\widehat{\mathcal{H}}_n^{\mathbb{K}}} M = \bigoplus_{a \in \mathbb{K}^*} e_a M.$$

We denote the socle of the $\widehat{\mathcal{H}}_{n-1}^{\mathbb{K}}$ -module $e_a M$ by

$$\tilde{e}_a M := \text{Soc}(e_a M).$$

The following modular branching rule for $\widehat{\mathcal{H}}_n^{\mathbb{K}}$ is a result of Grojnowski-Vazirani.

Proposition 4.3. (See [GV, Theorems (A) and (B)].) *Let M be a simple $\widehat{\mathcal{H}}_n^{\mathbb{K}}$ -module and $a \in \mathbb{K}^*$. Then either $\tilde{e}_a M = 0$ or $\tilde{e}_a M$ is simple. Moreover, the socle of $\text{Res}_{\widehat{\mathcal{H}}_{n-1}^{\mathbb{K}}}^{\widehat{\mathcal{H}}_n^{\mathbb{K}}} M$ is multiplicity free.*

We start with a preparatory result.

Lemma 4.4. *Suppose that $\mu = (\mu_1, \dots, \mu_r) \in \mathcal{C}_r(n)$ and let L_k ($1 \leq k \leq r$) be a $\widehat{\mathcal{H}}_{\mu_k}^{\mathbb{K}}$ -module. Then*

$$\begin{aligned} & \text{Ind}_{\widehat{Y}_{r,\mu}^{\mathbb{K}}}^{\widehat{Y}_{r,n}^{\mathbb{K}}} ((V_1^{\otimes \mu_1} \otimes L_1) \otimes \dots \otimes (V_r^{\otimes \mu_r} \otimes L_r)) \\ & \cong \text{Ind}_{\widehat{Y}_{r,\tau(\mu)}^{\mathbb{K}}}^{\widehat{Y}_{r,n}^{\mathbb{K}}} ((V_{\tau(1)}^{\otimes \mu_{\tau(1)}} \otimes L_{\tau(1)}) \otimes \dots \otimes (V_{\tau(r)}^{\otimes \mu_{\tau(r)}} \otimes L_{\tau(r)})), \end{aligned}$$

where $\tau(\mu) = (\mu_{\tau(1)}, \dots, \mu_{\tau(r)})$ for any $\tau \in \mathfrak{S}_r$.

Proof. We denote the left-hand side and the right-hand side of this isomorphism in the lemma by L and R , respectively. By Theorem 3.7, it suffices to show that $\mathcal{F}(L) \cong \mathcal{F}(R)$. Indeed, for any $\nu \neq \mu \in \mathcal{C}_r(n)$, $\text{Hom}_{\mathbb{K}T}(V(\nu), I_\nu L) = \text{Hom}_{\mathbb{K}T}(V(\nu), I_\nu R) = 0$ (actually $I_\nu L = I_\nu R = 0$). We also have the next isomorphism

$$\text{Hom}_{\mathbb{K}T}(V(\mu), I_\mu L) \cong L_1 \otimes \dots \otimes L_r \cong \text{Hom}_{\mathbb{K}T}(V(\mu), I_\mu R).$$

Thus, we have proved this lemma. \square

Given an r -composition $\mu = (\mu_1, \dots, \mu_r) \in \mathcal{C}_r(n)$, we denote by

$$\mu_i^- = (\mu_1, \dots, \mu_i - 1, \dots, \mu_r), \quad \mu_i^+ = (\mu_1, \dots, \mu_i + 1, \dots, \mu_r)$$

the r -compositions of $n \mp 1$ associated with μ for $1 \leq i \leq r$. (It is understood that the terms involving μ_i^- disappear for those i with $\mu_i = 0$.)

Recall that $\widehat{Y}_{r,n-1,1}^{\mathbb{K}}$ is the subalgebra of $\widehat{Y}_{r,n}^{\mathbb{K}}$ generated by $\mathbb{K}T$, $X_1^{\pm 1}, \dots, X_n^{\pm 1}$, and g_w for all $w \in \mathfrak{S}_{n-1}$. Then we have $\widehat{Y}_{r,n-1,1}^{\mathbb{K}} \cong \widehat{Y}_{r,n-1}^{\mathbb{K}} \otimes \widehat{Y}_{r,1}^{\mathbb{K}}$. The following result can be considered as a variant of Mackey's lemma, and the L_k ($1 \leq k \leq r$) in $S_\mu(L)$ are not necessarily simple modules.

Lemma 4.5. *Suppose that $\mu = (\mu_1, \dots, \mu_r) \in \mathcal{C}_r(n)$ and L_k ($1 \leq k \leq r$) is a $\widehat{\mathcal{H}}_{\mu_k}^{\mathbb{K}}$ -module. Then we have*

$$\text{Res}_{\widehat{Y}_{r,n-1,1}^{\mathbb{K}}}^{\widehat{Y}_{r,n}^{\mathbb{K}}} S_\mu(L) \cong \bigoplus_{a \in \mathbb{K}^*, 1 \leq k \leq r} S_{\mu_k^-}(e_a L) \otimes (V_k \otimes L(a)),$$

where $L(a)$ is the one-dimensional $\mathbb{K}[X^{\pm 1}]$ -module with $X^{\pm 1}$ acting as the scalar $a^{\pm 1}$ and $S_{\mu_k^-}(e_a L)$ denotes the $\widehat{Y}_{r,n-1}^{\mathbb{K}}$ -module

$$\text{Ind}_{\widehat{Y}_{r,\mu_k^-}^{\mathbb{K}}}^{\widehat{Y}_{r,n-1}^{\mathbb{K}}} ((V_1^{\otimes \mu_1} \otimes L_1) \otimes \dots \otimes (V_k^{\otimes (\mu_k - 1)} \otimes e_a L_k) \otimes \dots \otimes (V_r^{\otimes \mu_r} \otimes L_r)).$$

Proof. It can be easily checked that $S_{\mu_k^-}(e_a L) \otimes (V_k \otimes L(a))$ is a $\widehat{Y}_{r,n-1,1}^{\mathbb{K}}$ -submodule of $\text{Res}_{\widehat{Y}_{r,n-1,1}^{\mathbb{K}}}^{\widehat{Y}_{r,n}^{\mathbb{K}}} S_\mu(L)$ for all $a \in \mathbb{K}^*$ by Mackey's lemma. If $\mu_r = 0$, it means that we take the biggest k satisfying $\mu_k \neq 0$. Then Lemma 4.4 implies that $S_{\mu_k^-}(e_a L) \otimes (V_k \otimes L(a))$ is a $\widehat{Y}_{r,n-1,1}^{\mathbb{K}}$ -submodule of $\text{Res}_{\widehat{Y}_{r,n-1,1}^{\mathbb{K}}}^{\widehat{Y}_{r,n}^{\mathbb{K}}} S_\mu(L)$ for each $a \in \mathbb{K}^*$ and $1 \leq k \leq r$, and hence we have

$$\sum_{a \in \mathbb{K}^*, 1 \leq k \leq r} S_{\mu_k^-}(e_a L) \otimes (V_k \otimes L(a)) \subseteq \text{Res}_{\widehat{Y}_{r,n-1,1}^{\mathbb{K}}}^{\widehat{Y}_{r,n}^{\mathbb{K}}} S_\mu(L).$$

Since $V_k \otimes L(a)$ are pairwise non-isomorphic simple $\widehat{Y}_{r,1}^{\mathbb{K}}$ -modules for distinct (k, a) , the above sum is indeed a direct sum, and then this lemma follows from a dimension comparison. \square

We are now ready to establish the modular branching rules for $\widehat{Y}_{r,n}^{\mathbb{K}}$.

Theorem 4.6. *Consider the simple $\widehat{Y}_{r,n}^{\mathbb{K}}$ -module $S_\mu(L)$ defined in Theorem 4.1. Then we have*

$$\text{Soc}(\text{Res}_{\widehat{Y}_{r,n-1,1}^{\mathbb{K}}}^{\widehat{Y}_{r,n}^{\mathbb{K}}} S_\mu(L)) \cong \bigoplus_{a \in \mathbb{K}^*, 1 \leq k \leq r} S_{\mu_k^-}(\tilde{e}_a L) \otimes (V_k \otimes L(a)),$$

where $S_{\mu_k^-}(\tilde{e}_a L)$ denotes the nonzero simple $\widehat{Y}_{r,n-1}^{\mathbb{K}}$ -module

$$\text{Ind}_{\widehat{Y}_{r,\mu_k^-}^{\mathbb{K}}}^{\widehat{Y}_{r,n-1}^{\mathbb{K}}} ((V_1^{\otimes \mu_1} \otimes L_1) \otimes \dots \otimes (V_k^{\otimes (\mu_k - 1)} \otimes \tilde{e}_a L_k) \otimes \dots \otimes (V_r^{\otimes \mu_r} \otimes L_r)).$$

Proof. It follows from Lemma 4.5 by observing that the socle of the $\widehat{Y}_{r,n-1}^{\mathbb{K}}$ -module $S_{\mu_k^-}(e_a L.)$ is $S_{\mu_k^-}(\tilde{e}_a L.)$. \square

4.3. A block decomposition. In this subsection, we will construct a decomposition of a module M in $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$, which is similar to [Kle2, Sections 4.1 and 4.2].

For any $\underline{s} = (s_1, \dots, s_n) \in (\mathbb{K}^*)^n$, let $M_{\underline{s}}$ be the simultaneous generalized eigenspace of M for the commuting invertible operators X_1, \dots, X_n with eigenvalues s_1, \dots, s_n . Then as a $P_n^{\mathbb{K}}$ -module, we have

$$M = \bigoplus_{\underline{s} \in (\mathbb{K}^*)^n} M_{\underline{s}}.$$

A given $\underline{s} \in (\mathbb{K}^*)^n$ defines a one-dimensional representation of the algebra $\Lambda_n = \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]^{\mathfrak{S}_n}$ as

$$\omega_{\underline{s}} : \Lambda_n \rightarrow \mathbb{K}, \quad f(X_1^{\pm 1}, \dots, X_n^{\pm 1}) \mapsto f(s_1^{\pm 1}, \dots, s_n^{\pm 1}).$$

Write $\underline{s} \sim \underline{t}$ if they lie in the same \mathfrak{S}_n -orbit. Observe that $\underline{s} \sim \underline{t}$ if and only if $\omega_{\underline{s}} = \omega_{\underline{t}}$. For each orbit $\gamma \in (\mathbb{K}^*)^n / \sim$, we set $\omega_{\gamma} := \omega_{\underline{s}}$ for any $\underline{s} \in \gamma$. Let

$$M[\gamma] = \{m \in M \mid (z - \omega_{\gamma}(z))^N m = 0 \text{ for all } z \in \Lambda_n \text{ and } N \gg 0\}.$$

Then we have

$$M[\gamma] = \bigoplus_{\underline{s} \in \gamma} M_{\underline{s}}.$$

Since Λ_n is contained in the center of $\widehat{Y}_{r,n}^{\mathbb{K}}$ by Theorem 2.7, $M[\gamma]$ is a $\widehat{Y}_{r,n}^{\mathbb{K}}$ -module and we have the following decomposition in $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$:

$$M = \bigoplus_{\gamma \in (\mathbb{K}^*)^n / \sim} M[\gamma]. \quad (4.1)$$

Recall the decomposition in Lemma 3.3. We set, for each $\mu \in \mathcal{C}_r(n)$ and $\gamma \in (\mathbb{K}^*)^n / \sim$, that $M[\mu, \gamma] := M_{\mu} \cap M[\gamma]$. Since $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ commute with $\mathbb{K}T$, it follows that $M[\mu, \gamma] = (M_{\mu})[\gamma] = (M[\gamma])_{\mu}$. Thus, combining Lemma 3.3 and (4.1), we have the following decomposition in $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$:

$$M = \bigoplus_{\mu \in \mathcal{C}_r(n), \gamma \in (\mathbb{K}^*)^n / \sim} M[\mu, \gamma]. \quad (4.2)$$

This gives us a block decomposition of $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$ by applying Theorem 3.7 and the well-known block decomposition for $\widehat{\mathcal{H}}_n$ over an algebraically closed field; see [Gr, Proposition 4.4] and also [LM, Theorem 2.15].

5. CYCLOTOMIC YOKONUMA-HECKE ALGEBRAS AND MORITA EQUIVALENCES

In this section, we establish an explicit equivalence between the category $Y_{r,n}^{\lambda, \mathbb{K}}\text{-mod}$ of finite dimensional $Y_{r,n}^{\lambda, \mathbb{K}}$ -modules and the category $\mathcal{H}_{r,n}^{\lambda, \mathbb{K}}\text{-mod}$ of finite dimensional $\mathcal{H}_{r,n}^{\lambda, \mathbb{K}}$ -modules, where $\mathcal{H}_{r,n}^{\lambda, \mathbb{K}}$ is a direct sum of tensor products for various cyclotomic Hecke algebras $\mathcal{H}_{\mu_i}^{\lambda, \mathbb{K}}$. This category equivalence plays a crucial role in Section 6.

5.1. Cyclotomic Yokonuma-Hecke algebras. Recall that $\mathbb{I} = \{q^i \mid i \in \mathbb{Z}\}$. A $\widehat{Y}_{r,n}^{\mathbb{K}}$ -module is called integral if it is finite dimensional and all eigenvalues of X_1, \dots, X_n on M belong to the set \mathbb{I} . We denote by $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}_{\mathbb{I}}$ the full subcategory of $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$ consisting of all integral $\widehat{Y}_{r,n}^{\mathbb{K}}$ -modules. Similarly, we can define integral $\widehat{\mathcal{H}}_n^{\mathbb{K}}$ -modules and the category $\widehat{\mathcal{H}}_n^{\mathbb{K}}\text{-mod}_{\mathbb{I}}$. It is explained in [Va, Remark 1] that to understand $\widehat{\mathcal{H}}_n^{\mathbb{K}}\text{-mod}$, it is enough to understand $\widehat{\mathcal{H}}_n^{\mathbb{K}}\text{-mod}_{\mathbb{I}}$, that is, the study of simple modules for $\widehat{\mathcal{H}}_n^{\mathbb{K}}$ can be reduced to that of integral simple $\widehat{\mathcal{H}}_n^{\mathbb{K}}$ -modules. Then by Theorem 3.7, to study simple $\widehat{Y}_{r,n}^{\mathbb{K}}$ -modules, it suffices to study simple objects in $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}_{\mathbb{I}}$.

Now we introduce the following intertwining elements in $\widehat{Y}_{r,n}^{\mathbb{K}}$:

$$\Theta_i := qg_i(1 - X_i X_{i+1}^{-1}) + (1 - q^2)e_i, \quad 1 \leq i \leq n-1.$$

Lemma 5.1. *For each $1 \leq i \leq n-1$, we have*

$$\Theta_i^2 = (1 - q^2)^2(e_i - 1) + (1 - q^2 X_i X_{i+1}^{-1})(1 - q^2 X_{i+1} X_i^{-1}); \quad (5.1)$$

$$\Theta_i X_i = X_{i+1} \Theta_i, \quad \Theta_i X_{i+1} = X_i \Theta_i, \quad \Theta_i X_j = X_j \Theta_i \text{ for } j \neq i, i+1. \quad (5.2)$$

Proof. By (2.11), we can prove these identities by a direct computation as follows.

$$\begin{aligned} \Theta_i^2 &= [qg_i(1 - X_i X_{i+1}^{-1}) + (1 - q^2)e_i]^2 \\ &= q^2 g_i(1 - X_i X_{i+1}^{-1})g_i(1 - X_i X_{i+1}^{-1}) + 2q(1 - q^2)g_i e_i(1 - X_i X_{i+1}^{-1}) + (1 - q^2)^2 e_i^2 \\ &= q^2 [1 + (q - q^{-1})e_i g_i](1 - X_i X_{i+1}^{-1}) - q^2 g_i X_i [g_i X_i^{-1} - (q - q^{-1})e_i X_i^{-1}] \\ &\quad \times (1 - X_i X_{i+1}^{-1}) + 2q(1 - q^2)g_i e_i(1 - X_i X_{i+1}^{-1}) + (1 - q^2)^2 e_i \\ &= q^2(1 - X_i X_{i+1}^{-1}) + q(q^2 - 1)g_i e_i(1 - X_i X_{i+1}^{-1}) - q^2 X_{i+1} X_i^{-1}(1 - X_i X_{i+1}^{-1}) \\ &\quad + q(q^2 - 1)g_i e_i(1 - X_i X_{i+1}^{-1}) + 2q(1 - q^2)g_i e_i(1 - X_i X_{i+1}^{-1}) + (1 - q^2)^2 e_i \\ &= (1 - q^2)^2(e_i - 1) + (1 - q^2 X_i X_{i+1}^{-1})(1 - q^2 X_{i+1} X_i^{-1}). \end{aligned}$$

$$\begin{aligned} \Theta_i X_i &= [qg_i(1 - X_i X_{i+1}^{-1}) + (1 - q^2)e_i]X_i \\ &= q[X_{i+1}g_i - (q - q^{-1})e_i X_{i+1}](1 - X_i X_{i+1}^{-1}) + (1 - q^2)e_i X_i \\ &= qX_{i+1}g_i(1 - X_i X_{i+1}^{-1}) - (q^2 - 1)e_i X_{i+1} + (q^2 - 1)e_i X_i + (1 - q^2)e_i X_i \\ &= X_{i+1}[qg_i(1 - X_i X_{i+1}^{-1}) + (1 - q^2)e_i] \\ &= X_{i+1}\Theta_i. \end{aligned}$$

$$\begin{aligned} \Theta_i X_{i+1} &= [qg_i(1 - X_i X_{i+1}^{-1}) + (1 - q^2)e_i]X_{i+1} \\ &= q[X_i g_i + (q - q^{-1})e_i X_{i+1}](1 - X_i X_{i+1}^{-1}) + (1 - q^2)e_i X_{i+1} \\ &= qX_i g_i(1 - X_i X_{i+1}^{-1}) + (q^2 - 1)e_i X_{i+1} - (q^2 - 1)e_i X_i + (1 - q^2)e_i X_{i+1} \\ &= X_i[qg_i(1 - X_i X_{i+1}^{-1}) + (1 - q^2)e_i] \\ &= X_i \Theta_i. \end{aligned}$$

By (2.9) and (2.10), we have $\Theta_i X_j = X_j \Theta_i$ for $j \neq i, i+1$. □

Lemma 5.2. *Let $M \in \widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$ and fix i with $1 \leq i \leq n$. Assume that all eigenvalues of X_i on M belong to \mathbb{I} . Then M is integral.*

Proof. It suffices to show that the eigenvalues of X_k on M belong to \mathbb{I} if and only if the eigenvalues of X_{k+1} on M belong to \mathbb{I} for $1 \leq k \leq n-1$. By Lemmas 3.2 and 3.3, it suffices to consider the subspaces $I_\mu M$ for all $\mu \in \mathcal{C}_r(n)$. Assume that all eigenvalues of X_{k+1} on $I_\mu M$ belong to \mathbb{I} . Let a be an eigenvalue for the action of X_k on $I_\mu M$. Since X_k and X_{k+1} commute, we can pick u lying in the a -eigenspace of X_k so that u is also an eigenvector for X_{k+1} , of eigenvalue b . By assumption, we have $b = q^s$ for some $s \in \mathbb{Z}$. By (5.2), we have $X_{k+1}\Theta_k = \Theta_k X_k$. So if $\Theta_k u \neq 0$, then we get that $X_{k+1}\Theta_k u = a\Theta_k u$; hence a is an eigenvalue of X_{k+1} , and so $a \in \mathbb{I}$ by assumption. Else, $\Theta_k u = 0$, then applying (5.1), we have

$$(1 - q^2)^2(e_k - 1)u + (1 - q^{2-s}a)(1 - q^{2+s}a^{-1})u = 0.$$

Since $I_\mu M$ is isomorphic to the direct sum of copies of $V_1^{\otimes \mu_1} \otimes \cdots \otimes V_r^{\mu_r}$, by Lemma 3.1, we have $e_k u = 0$ or $e_k u = u$. Thus, we must have $a = q^s$ or $a = q^{s \pm 2}$. We again have $a \in \mathbb{I}$. Similarly, we can show that all eigenvalues of X_{k+1} on $I_\mu M$ belong to \mathbb{I} if we assume all eigenvalues of X_k on $I_\mu M$ belong to \mathbb{I} . \square

Set $\mathbb{J} = \{0, 1, \dots, e-1\}$, where e is the order of $q \in \mathbb{K}^*$. Let

$$\Delta := \{\lambda = (\lambda_i)_{i \in \mathbb{J}} \mid \lambda_i \in \mathbb{Z}_{\geq 0} \text{ and only finitely many } \lambda_i \text{ are nonzero}\}.$$

Let

$$f_\lambda \equiv f_\lambda(X_1) = \prod_{i \in \mathbb{J}} (X_1 - q^i)^{\lambda_i}.$$

The cyclotomic Yokonuma-Hecke algebra $Y_{r,n}^{\lambda, \mathbb{K}}$ is defined to be the quotient algebra by the two-sided ideal \mathcal{J}_λ of $\widehat{Y}_{r,n}^{\mathbb{K}}$ generated by f_λ , that is,

$$Y_{r,n}^{\lambda, \mathbb{K}} = \widehat{Y}_{r,n}^{\mathbb{K}} / \mathcal{J}_\lambda, \quad \lambda \in \Delta.$$

Lemma 5.3. *Let $M \in \widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$. Then M is integral if and only if $\mathcal{J}_\lambda M = 0$ for some $\lambda \in \Delta$.*

Proof. If $\mathcal{J}_\lambda M = 0$, then the eigenvalue of X_1 on M are all in \mathbb{I} . Hence M is integral by Lemma 5.2. Conversely, suppose that M is integral. Then the minimal polynomial of X_1 on M is of the form $\prod_{i \in \mathbb{J}} (t - q^i)^{\lambda_i}$ for some $\lambda_i \in \mathbb{Z}_{\geq 0}$. So if we set \mathcal{J}_λ to be the two-sided ideal of $\widehat{Y}_{r,n}^{\mathbb{K}}$ generated by $\prod_{i \in \mathbb{J}} (X_1 - q^i)^{\lambda_i}$, we certainly have that $\mathcal{J}_\lambda M = 0$. \square

By inflation along the canonical homomorphism $\widehat{Y}_{r,n}^{\mathbb{K}} \rightarrow Y_{r,n}^{\lambda, \mathbb{K}}$, we can identify $Y_{r,n}^{\lambda, \mathbb{K}}\text{-mod}$ with the full subcategory of $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$ consisting of all modules M with $\mathcal{J}_\lambda M = 0$. By Lemma 5.3, to study modules in the category $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}_{\mathbb{I}}$, we may instead study modules in the category $Y_{r,n}^{\lambda, \mathbb{K}}\text{-mod}$ for all $\lambda \in \Delta$.

The next proposition follows from [ChP2, Theorem 4.4]. In fact, we can adapt all the claims in [Kle2, Section 7.5] to our setting and give a direct proof of the PBW basis theorem for $Y_{r,n}^{\lambda, \mathbb{K}}$; see [C, Section 2] for more details.

Proposition 5.4. *Suppose $\lambda \in \Delta$. Let $d = |\lambda| = \sum_{i \in \mathbb{J}} \lambda_i$. The following elements*

$$\{X^\alpha t g_w \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n \text{ with } 0 \leq \alpha_1, \dots, \alpha_n \leq d-1, t \in T, w \in \mathfrak{S}_n\}$$

form a basis for $Y_{r,n}^{\lambda, \mathbb{K}}$.

5.2. The functors e_{j, χ^k}^λ and f_{j, χ^k}^λ . In view of (4.2), we have the following decomposition in $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}_{\mathbb{I}}$:

$$M = \bigoplus_{\mu \in \mathcal{C}_r(n), \gamma \in \mathbb{I}^n / \sim} M[\mu, \gamma].$$

Set Γ_n to be the set of nonnegative integral linear combinations $\gamma = \sum_{j \in \mathbb{J}} \gamma_j \varepsilon_j$ of the standard basis ε_j of $\mathbb{Z}^{|\mathbb{J}|}$ such that $\sum_{j \in \mathbb{J}} \gamma_j = n$. If $\underline{s} \in \mathbb{I}^n$, we define its content by

$$\text{cont}(\underline{s}) := \sum_{j \in \mathbb{J}} \gamma_j \varepsilon_j \in \Gamma_n, \text{ where } \gamma_j = \#\{k = 1, 2, \dots, n \mid s_k = q^j\}.$$

The content function induces a canonical bijection between \mathbb{I}^n / \sim and Γ_n , and we will identify the two sets. Now the above decomposition in $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}_{\mathbb{I}}$ can be rewritten as

$$M = \bigoplus_{\mu \in \mathcal{C}_r(n), \gamma \in \Gamma_n} M[\mu, \gamma]. \quad (5.3)$$

Such a decomposition also makes sense in the category $Y_{r,n}^{\lambda, \mathbb{K}}\text{-mod}$.

Observe that the subalgebra of $Y_{r,n}^{\lambda, \mathbb{K}}$ generated by $X_1^{\pm 1}, \dots, X_{n-1}^{\pm 1}$, $\mathbb{K}T$, and g_w for all $w \in \mathfrak{S}_{n-1}$ is isomorphic to $Y_{r,n-1}^{\lambda, \mathbb{K}} \otimes \mathbb{K}G$ by Proposition 5.4.

Definition 5.5. Suppose that $M \in Y_{r,n}^{\lambda, \mathbb{K}}\text{-mod}$ and that $M = M[\mu, \gamma]$ for some $\mu \in \mathcal{C}_r(n)$ and $\gamma \in \Gamma_n$. For each $j \in \mathbb{J}$ and $1 \leq k \leq r$, we define

$$e_{j, \chi^k}^\lambda M = \text{Hom}_{\mathbb{K}G}(V_k, \text{Res}_{Y_{r,n-1}^{\lambda, \mathbb{K}} \otimes \mathbb{K}G}^{Y_{r,n}^{\lambda, \mathbb{K}}} M) [\mu_k^-, \gamma - \varepsilon_j],$$

$$f_{j, \chi^k}^\lambda M = \left(\text{Ind}_{Y_{r,n}^{\lambda, \mathbb{K}} \otimes \mathbb{K}G}^{Y_{r,n+1}^{\lambda, \mathbb{K}}} (M \otimes V_k) \right) [\mu_k^+, \gamma + \varepsilon_j].$$

We extend e_{j, χ^k}^λ (resp. f_{j, χ^k}^λ) to functors from $Y_{r,n}^{\lambda, \mathbb{K}}\text{-mod}$ to $Y_{r,n-1}^{\lambda, \mathbb{K}}\text{-mod}$ (resp. from $Y_{r,n}^{\lambda, \mathbb{K}}\text{-mod}$ to $Y_{r,n+1}^{\lambda, \mathbb{K}}\text{-mod}$) by the direct sum decomposition (5.3).

Remark 5.6. When $r = 1$, the functors e_{j, χ^k}^λ and f_{j, χ^k}^λ (with the index χ^k dropped) coincide with the ones e_j^λ and f_j^λ defined by Ariki and Grojnowski; see [Ari1] and [Gr].

5.3. A Morita equivalence. Let \mathfrak{S}'_{n-1} be the subgroup of \mathfrak{S}_n generated by s_2, \dots, s_{n-1} . For each $\mu = (\mu_1, \dots, \mu_r) \in \mathcal{C}_r(n)$ and $1 \leq k \leq r$, we set $\bar{\mu}^k = \mu_1 + \dots + \mu_k$. The next lemma follows from [Ze, Proposition A.3.2].

Lemma 5.7. (See [WW, Lemma 5.10].) *There exists a complete set $\mathcal{O}(\mu)$ of left coset representatives of \mathfrak{S}_μ in \mathfrak{S}_n such that any $w \in \mathcal{O}(\mu)$ is of the form $\sigma(1, \bar{\mu}^k + 1)$ for some $\sigma \in \mathfrak{S}'_{n-1}$ and $0 \leq k \leq r-1$. (It is understood that $(1, \bar{\mu}^k + 1) = 1$ if $k = 0$.)*

Note that $(1, m+1) = s_m \cdots s_2 s_1 s_2 \cdots s_m$. By (2.12) and the identity $e_{i,j} g_j = g_j e_{i,j+1}$ for $1 \leq i < j \leq n-1$ in $\widehat{Y}_{r,n}^{\mathbb{K}}$, we can easily get the following result.

Lemma 5.8. *Let $\mu \in \mathcal{C}_r(n)$. Fix $0 \leq k \leq r-1$ and let $w_\mu^k = (1, \bar{\mu}^k + 1)$. Then we have*

$$X_1 g_{w_\mu^k} = g_{w_\mu^k} X_{\bar{\mu}^k+1} - (q - q^{-1}) \sum_{l=1}^{\bar{\mu}^k} g_{\bar{\mu}^k} \cdots g_2 g_1 g_2 \cdots \widehat{g_l^{X_{l+1}}} \cdots g_{\bar{\mu}^k} e_{l, \bar{\mu}^k+1},$$

where $\widehat{g_l^{X_{l+1}}}$ means replacing g_l with X_{l+1} .

Let $\{\alpha_i \mid i \in \mathbb{J}\}$ be the simple roots of the affine Lie algebra \widehat{sl}_e and $\{h_i \mid i \in \mathbb{J}\}$ be the corresponding simple coroots. Let P_+ be the set of all dominant integral weights. For each $\mu \in P_+$, we define the cyclotomic Hecke algebra \mathcal{H}_n^μ by

$$\mathcal{H}_n^\mu = \widehat{\mathcal{H}}_n / \left\langle \prod_{i \in \mathbb{J}} (Y_i - q^i)^{\langle h_i, \mu \rangle} \right\rangle.$$

We set $\mathcal{H}_n^{\mu, \mathbb{K}} = \mathbb{K} \otimes_{\mathcal{R}} \mathcal{H}_n^\mu$.

For each $\lambda \in \Delta$, we define $\lambda' \in P_+$ by $\langle h_i, \lambda' \rangle = \lambda_i, \forall i \in \mathbb{J}$. Thus, we have a one-to-one correspondence between Δ and P_+ , and we will identify the two sets. Furthermore, we define the following algebra:

$$\mathcal{H}_{r,n}^{\lambda, \mathbb{K}} = \bigoplus_{\mu \in \mathcal{C}_r(n)} \mathcal{H}_{\mu_1}^{\lambda, \mathbb{K}} \otimes \cdots \otimes \mathcal{H}_{\mu_r}^{\lambda, \mathbb{K}}.$$

Recall the functor \mathcal{F} defined in Theorem 3.7. Then we have the following result.

Theorem 5.9. *\mathcal{F} induces a category equivalence $\mathcal{F}^\lambda : Y_{r,n}^{\lambda, \mathbb{K}}\text{-mod} \rightarrow \mathcal{H}_{r,n}^{\lambda, \mathbb{K}}\text{-mod}$.*

Proof. The category $Y_{r,n}^{\lambda, \mathbb{K}}\text{-mod}$ can be identified with the full subcategory of $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$ consisting of all modules M with $\mathcal{J}_\lambda M = 0$. By Lemma 3.3, $\mathcal{J}_\lambda M = 0$ if and only if $\mathcal{J}_\lambda M_\mu = 0$ for each $\mu \in \mathcal{C}_r(n)$. By Lemma 3.2 and Proposition 3.6, we have

$$M_\mu \cong \text{Ind}_{\widehat{Y}_{r,\mu}^{\mathbb{K}}}^{\widehat{Y}_{r,n}^{\mathbb{K}}}(I_\mu M), \quad I_\mu M \cong V(\mu) \otimes_{\mathbb{K}} \text{Hom}_{\mathbb{K}T}(V(\mu), I_\mu M).$$

As vector spaces, we have

$$M_\mu = \bigoplus_{w \in \mathcal{O}(\mu)} g_w \otimes I_\mu M.$$

By Lemma 5.7, for each $w \in \mathcal{O}(\mu)$, there exists $\sigma \in \mathfrak{S}'_{n-1}$ such that $w = \sigma(1, \bar{\mu}^k + 1) = \sigma w_\mu^k$ for some $0 \leq k \leq r-1$. Note that $e_{l, \bar{\mu}^k+1} = 0$ on $I_\mu M$ for $1 \leq l \leq \bar{\mu}^k$. So we have

$$X_1 g_{w_\mu^k} \otimes z = g_{w_\mu^k} \otimes X_{\bar{\mu}^k+1} z$$

for $z \in I_\mu M$ by Lemma 5.8, and thus $f_\lambda g_w \otimes z = g_w \otimes f_{\lambda,k} z$, where

$$f_{\lambda,k} := \prod_{i \in \mathbb{J}} (X_{\bar{\mu}^k+1} - q^i)^{\lambda_i}.$$

Therefore, $f_\lambda M_\mu = 0$ if and only if $f_{\lambda,k} I_\mu M = 0$ for all $0 \leq k \leq r-1$. By Propositions 3.4-3.6, $f_{\lambda,k}$ acts as zero on $I_\mu M$ if and only if $\prod_{i \in \mathbb{J}} (Y_{\bar{\mu}^k+1} - q^i)^{\lambda_i}$ acts as zero on

$\text{Hom}_{\mathbb{K}T}(V(\mu), I_\mu M)$. Therefore, $f_\lambda M = 0$ if and only if $\text{Hom}_{\mathbb{K}T}(V(\mu), I_\mu M) \in \mathcal{H}_{r,n}^{\lambda, \mathbb{K}}\text{-mod}$ for each $\mu \in \mathcal{C}_r(n)$ as desired. \square

6. APPLICATIONS

In this section, we will present several applications of the category equivalence obtained in the preceding section. We shall classify all finite dimensional simple $Y_{r,n}^{\lambda, \mathbb{K}}$ -modules, and establish the modular branching rule for $Y_{r,n}^{\lambda, \mathbb{K}}$ which provides a description of the socle of the restriction to $Y_{r,n-1,1}^{\lambda, \mathbb{K}}$ of a simple $Y_{r,n}^{\lambda, \mathbb{K}}$ -module. We also give a crystal graph interpretation for modular branching rules. In the end, we will give a block decomposition of $Y_{r,n}^{\lambda, \mathbb{K}}\text{-mod}$.

6.1. The simple $Y_{r,n}^{\lambda, \mathbb{K}}$ -modules. Let $\text{ev}_{n,\lambda}$ denote the surjective algebra homomorphism $\text{ev}_{n,\lambda} : \widehat{\mathcal{H}}_n^{\mathbb{K}} \rightarrow \mathcal{H}_n^{\lambda, \mathbb{K}}$ for any n . Then an $\mathcal{H}_n^{\lambda, \mathbb{K}}$ -module L can be regarded as an $\widehat{\mathcal{H}}_n^{\mathbb{K}}$ -module by inflation, denoted by $\text{ev}_{n,\lambda}^* L$. From the proof of Theorem 5.9, we see that if L_k ($1 \leq k \leq r$) is a simple $\mathcal{H}_{\mu_k}^{\lambda, \mathbb{K}}$ -module, then $S_\mu(L.)$ is in fact a $Y_{r,n}^{\lambda, \mathbb{K}}$ -module. Thus, by Theorem 4.1, we immediately get the following result.

Theorem 6.1. *Each simple $Y_{r,n}^{\lambda, \mathbb{K}}$ -module is isomorphic to a module of the form*

$$S_\mu(L.) := \text{Ind}_{\widehat{\mathcal{H}}_{r,\mu}^{\mathbb{K}}}^{\widehat{\mathcal{H}}_{r,n}^{\mathbb{K}}} ((V_1^{\otimes \mu_1} \otimes \text{ev}_{\mu_1, \lambda}^* L_1) \otimes \cdots \otimes (V_r^{\otimes \mu_r} \otimes \text{ev}_{\mu_r, \lambda}^* L_r)),$$

where $\mu = (\mu_1, \dots, \mu_r) \in \mathcal{C}_r(n)$, and L_k ($1 \leq k \leq r$) is a simple $\mathcal{H}_{\mu_k}^{\lambda, \mathbb{K}}$ -module. Moreover, the above modules $S_\mu(L.)$, for various $\mu \in \mathcal{C}_r(n)$ and L_k ($1 \leq k \leq r$), form a complete set of pairwise non-isomorphic simple $Y_{r,n}^{\lambda, \mathbb{K}}$ -modules.

Ariki has given the classification of simple modules of a cyclotomic Hecke algebra over an arbitrary field \mathbb{F} in terms of Kleshchev multipartitions. Let \mathcal{J}_n^λ be the set of all $|\lambda|$ -multipartitions of n . We denote by \mathcal{K}_n^λ the set of all Kleshchev multipartitions in \mathcal{J}_n^λ ; see [Ari2, Definition 2.3] for the precise definition. Set $\mathcal{H}_n^{\lambda, \mathbb{K}} = \mathbb{K} \otimes \mathcal{H}_n^\lambda$. Then the simple $\mathcal{H}_n^{\lambda, \mathbb{K}}$ -modules are parameterized by \mathcal{K}_n^λ ([Ari2, Theorem 4.2]).

Combining this fact with Theorem 6.1, we immediately obtain the next result.

Corollary 6.2. *The simple $Y_{r,n}^{\lambda, \mathbb{K}}$ -modules are parameterized by the set*

$$\mathcal{B} = \{(\mu, \psi_1, \dots, \psi_r) \mid \mu = (\mu_1, \dots, \mu_r) \in \mathcal{C}_r(n), \psi_i \in \mathcal{K}_{\mu_i}^\lambda, 1 \leq i \leq r\}.$$

In the case $|\lambda| = \sum_{i \in \mathbb{J}} \lambda_i = 1$, $Y_{r,n}^{\lambda, \mathbb{K}}$ is just the Yokonuma-Hecke algebra $Y_{r,n}^{\mathbb{K}}$, and \mathcal{K}_n^λ is exactly the set of e -restricted partitions of n (recall that e is the order of q in \mathbb{K}^*). Thus, we have also obtained the following corollary.

Corollary 6.3. *The simple $Y_{r,n}^{\mathbb{K}}$ -modules are parameterized by the set*

$$\mathcal{C} = \{(\mu, \psi_1, \dots, \psi_r) \mid \mu \in \mathcal{C}_r(n) \text{ and each } \psi_i \text{ is an } e\text{-restricted partition of } \mu_i\}.$$

Remark 6.4. The classification of simple modules of a Yokonuma-Hecke algebra in the split semisimple and non split semisimple case has been described in [JP, Section 4.1]. The simple modules of a cyclotomic Yokonuma-Hecke algebra in the generic semisimple case have been classified in [ChP2, Proposition 3.14].

6.2. Branching rules for $Y_{r,n}^{\lambda,\mathbb{K}}$ and a crystal graph interpretation. We denote by $K(\mathcal{A})$ the Grothendieck group of a module category \mathcal{A} and by $\text{Irr}(\mathcal{A})$ the set of pairwise non-isomorphic simple objects in \mathcal{A} . For each $\lambda \in P_+$, let

$$K(\lambda) = \bigoplus_{n \geq 0} K(\mathcal{H}_n^\lambda\text{-mod}), \quad K(\lambda)_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} K(\lambda).$$

Besides the functors e_i^λ and f_i^λ for \mathcal{H}_n^λ (see Remark 5.6), we define two additional operators \tilde{e}_i^λ and \tilde{f}_i^λ on $\coprod_{n \geq 0} \text{Irr}(\mathcal{H}_n^\lambda\text{-mod})$ by setting $\tilde{e}_i^\lambda L = \text{Soc}(e_i^\lambda L)$ and $\tilde{f}_i^\lambda L = \text{Head}(f_i^\lambda L)$ for each simple \mathcal{H}_n^λ -module L . We also have the operator \tilde{e}_i^λ on the set \mathcal{K}_n^λ (see [Ari3] for the definition of $\tilde{e}_i^\lambda \mu$).

Denote by $L(\lambda)$ the irreducible highest weight $\widehat{\mathfrak{sl}}_e$ -module of highest weight $\lambda \in P_+$. The next results are proved in [Ari1, Theorem 4.4] and [Gr, Theorem 12.3].

Proposition 6.5. *Let $\lambda \in P_+$. Then $K(\lambda)_{\mathbb{C}}$ is an $\widehat{\mathfrak{sl}}_e$ -module with the Chevalley generators acting as e_i^λ and f_i^λ ($i \in \mathbb{J}$); $K(\lambda)_{\mathbb{C}}$ is isomorphic to $L(\lambda)$ as $\widehat{\mathfrak{sl}}_e$ -modules.*

Moreover, $\coprod_{n \geq 0} \text{Irr}(\mathcal{H}_n^\lambda\text{-mod})$ is isomorphic to the crystal basis $B(\lambda)$ of the simple $\widehat{\mathfrak{sl}}_e$ -module $L(\lambda)$ with operators \tilde{e}_i^λ and \tilde{f}_i^λ identified with the Kashiwara operators.

We also have the next modular branching rules for cyclotomic Hecke algebras.

Proposition 6.6. (See [Ari3, Theorem 6.1].) *For each $\mu \in \mathcal{K}_n^\lambda$, let D^μ be the corresponding simple \mathcal{H}_n^λ -module. Then we have $\tilde{e}_i^\lambda D^\mu = D^{\tilde{e}_i^\lambda \mu}$.*

For each $\lambda \in \Delta$, let

$$K_T(\lambda) = \bigoplus_{n \geq 0} K(Y_{r,n}^{\lambda,\mathbb{K}}\text{-mod}), \quad K_T(\lambda)_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} K_T(\lambda).$$

Recall the functors e_{i,χ^k}^λ and f_{i,χ^k}^λ defined in Definition 5.5 for $i \in \mathbb{J}$ and $1 \leq k \leq r$. They induce linear operators on $K_T(\lambda)_{\mathbb{C}}$. By Theorem 5.9, the category equivalence \mathcal{F}^λ induces a canonical linear isomorphism

$$\mathcal{F}^\lambda : K_T(\lambda) \xrightarrow{\sim} K(\lambda) \otimes \cdots \otimes K(\lambda) \cong K(\lambda)^{\otimes r}. \quad (6.1)$$

We shall identify $Y_{r,n}^{\lambda,\mathbb{K}}\text{-mod}$ with a full subcategory of $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$. By Lemma 4.5, the functor e_{i,χ^k}^λ corresponds via \mathcal{F}^λ to e_i^λ applied to the k -th tensor factor on the right-hand side of (6.1). By Frobenius reciprocity, f_{i,χ^k}^λ is left adjoint to e_{i,χ^k}^λ and f_i^λ is left adjoint to e_i^λ ; hence f_{i,χ^k}^λ corresponds to f_i^λ applied to the k -th tensor factor on the right-hand side of (6.1). With the identification of $Y_{r,n}^{\lambda,\mathbb{K}}\text{-mod}$ with a full subcategory of $\widehat{Y}_{r,n}^{\mathbb{K}}\text{-mod}$, Theorem 4.6 and Proposition 6.6 implies the following modular branching rules for $Y_{r,n}^{\lambda,\mathbb{K}}$.

Theorem 6.7. *We have*

$$\text{Soc}(\text{Res}_{Y_{r,n-1,1}^{\lambda,\mathbb{K}}}^{Y_{r,n}^{\lambda,\mathbb{K}}} S_\mu(L)) \cong \bigoplus_{i \in \mathbb{J}, 1 \leq k \leq r} S_{\mu_k^-}(\tilde{e}_i^\lambda L) \otimes (V_k \otimes L(i)),$$

where $Y_{r,n-1,1}^{\lambda,\mathbb{K}}$ denotes the subalgebra of $Y_{r,n}^{\lambda,\mathbb{K}}$ generated by $X_1^{\pm 1}, \dots, X_n^{\pm 1}$, $\mathbb{K}T$, and g_w for all $w \in \mathfrak{S}_{n-1}$, and $L(i)$ is the one-dimensional $\mathbb{K}[X^{\pm 1}]$ -module with $X^{\pm 1}$ acting as the scalar $q^{\pm i}$.

Combining this with Theorem 5.9 and Proposition 6.5, we have established the following result.

Theorem 6.8. $K_T(\lambda)_{\mathbb{C}}$ affords a simple $\widehat{\mathfrak{sl}}_e^{\oplus r}$ -module isomorphic to $L(\lambda)^{\otimes r}$ with the Chevalley generators of the k -th summand of $\widehat{\mathfrak{sl}}_e^{\oplus r}$ acting as e_{i,χ^k}^λ and f_{i,χ^k}^λ ($i \in \mathbb{J}$) for each $1 \leq k \leq r$.

Moreover, $\coprod_{n \geq 0} \text{Irr}(Y_{r,n}^{\lambda,\mathbb{K}}\text{-mod})$ (and respectively, the modular branching graph given by Theorem 6.7) is isomorphic to the crystal basis $B(\lambda)^{\otimes r}$ (and respectively, the corresponding crystal graph) of the simple $\widehat{\mathfrak{sl}}_e^{\oplus r}$ -module $L(\lambda)^{\otimes r}$.

6.3. A block decomposition of $Y_{r,n}^{\lambda,\mathbb{K}}\text{-mod}$. The blocks of the cyclotomic Hecke algebra \mathcal{H}_n^λ over an arbitrary algebraically closed field have been classified in [LM, Theorem A]. By the Morita equivalence established in Theorem 5.9, the decomposition (5.3) provides us a block decomposition of $Y_{r,n}^{\lambda,\mathbb{K}}\text{-mod}$.

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